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13. ABSTRACT. We are concerned with the following reliability problem: A system has k different types of components. Associated with each component is a numerical value. Let $\{a^j\}$ $(j = 1,\ldots,k)$ denote the set of numerical values of the k components. Let $R(a^1,\ldots,a^k)$ denote the probability that the system will perform satisfactorily (i.e. $R(a^1,\ldots,a^k)$ is the reliability of the system) and assume $R(a^1,\ldots,a^k)$ has the properties of a joint cumulative distribution function.

Now suppose $a^j_1 \leq \ldots \leq a^j_n$ are n components of type j $(j = 1,\ldots,k)$. Then n systems can be assembled from these components. Let N denote the number of systems that perform satisfactorily. N is a random variable whose distribution will depend on the way the n systems are assembled. Of all different ways in which the n systems can be assembled, the paper shows that EN is maximized if these n systems have reliability $R(a^1_i,\ldots,a^k_i)$ $(i = 1,\ldots,n)$. The method used here is an extension of a well known result of Hardy, Littlewood and Polya on sums of products. Furthermore, under certain conditions, the same assembly that maximizes EN minimizes the variance of N.

Finally, for a similar problem in reliability, it is shown that for a series systems a construction can be found that not only maximizes the expected number of functioning modules but also possesses the stronger property of maximizing the probability that the number of functioning modules is at least r, for each $0 \leq r \leq n$.

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ON OPTIMAL ASSEMBLY OF SYSTEMS

by

Cyrus Derman, Gerald J. Lieberman & Sheldon M. Ross

Technical Report No. 140

August 25, 1971

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with the Office of Naval Research

Gerald J. Lieberman, Project Director

Department of Operations Research
&
Department of Statistics
Stanford University
Stanford, California

ON OPTIMAL ASSEMBLY OF SYSTEMS

by

Cyrus Derman, Gerald J. Lieberman & Sheldon M. Ross

[0] Summary

~~We are~~ concerned with the following reliability problem: A system has k different types of components. Associated with each component is a numerical value. Let $\{a^j\}$ $(j = 1,\ldots,k)$ denote the set of numerical values of the k components. Let $R(a^1,\ldots,a^k)$ denote the probability that the system will perform satisfactorily (i.e. $R(a^1,\ldots,a^k)$ is the reliability of the system) and assume $R(a^1,\ldots,a^k)$ has the properties of a joint cumulative distribution function.

Now suppose $a_1^j \leq \ldots \leq a_n^j$ are n components of type j $(j = 1,\ldots,k)$. Then n systems can be assembled from these components. Let N denote the number of systems that perform satisfactorily. N is a random variable whose distribution will depend on the way the n systems are assembled. Of all different ways in which the n systems can be assembled, the paper shows that EN is maximized if these n systems have reliability $R(a_i^1,\ldots,a_i^k)$ $(i = 1,\ldots,n)$. The method used here is an extension of a well known result of Hardy, Littlewood, and Polya on sums of products. Furthermore, under certain conditions, the same assembly that maximizes EN minimizes the variance of N.

Finally, for a similar problem in reliability, it is shown that for a series systems a construction can be found that not only

maximizes the expected number of functioning modules but also possesses the stronger property of maximizing the probability that the number of functioning modules is at least r, for each $0 \le r \le n$.

[1] An Optimal Assignment Theorem

A well-known result appearing in Hardy, Littlewood, and Polya [1] asserts if $a_1 \le a_2 \le \dots \le a_n$ and if $b_1 \le b_2 \le \dots \le b_n$, then

$$\sum_{i=1}^{n} a_i b_i \quad \ge \quad \sum_{i=1}^{n} a_i b_{\psi(i)}$$

where ψ is any permutation of the integers $1, 2, \dots, n$.

Also in [1] is the generalization that for any $k \ge 2$ and if $0 \le a_1^j \le \dots \le a_n^j$, $j = 1, \dots, k$ then

$$\sum_{i=1}^{n} \prod_{j=1}^{k} a_i^j \quad \ge \quad \sum_{i=1}^{n} \prod_{j=1}^{k} a_{\psi_j(i)}^j$$

where $\psi_1(i) = i$ $(i = 1, \dots, n)$ and $\psi_2, \dots, \psi_k$ are any k permutations of $1, 2, \dots, n$. Proof of both results follows from establishing the inequalities for the special case of $n = 2$ and then resorting to a standard argument regarding pairwise interchanges. First let $n = 2$ and $k = 2$. Then

$$a_1 b_1 + a_2 b_2 - a_1 b_2 - a_2 b_1 = a_1(b_1 - b_2) - a_2(b_1 - b_2) = (a_1 - a_2)(b_1 - b_2) \ge 0$$

since $a_1 \le a_2$ and $b_1 \le b_2$. Now assume the result is true for $n = 2$

and $k = 2, 3, \ldots, K-1$. Then consider

$$\sum_{i=1}^{2} \prod_{j=1}^{K} a^j_{\psi_j(i)}.$$

If for each $j = 2, \ldots, K$, $\psi_j(1) = 2$, $\psi_j(2) = 1$. Then

$$\prod_{j=2}^{K} a^j_{\psi_j(1)} = b_2 \qquad \text{(say)}$$

and

$$\begin{aligned} \prod_{j=2}^{K} a^j_{\psi_j(2)} &= b_1 \qquad \text{(say)} \\ &\leq b_2 \,, \end{aligned}$$

and, hence,

$$\begin{aligned} \prod_{j=1}^{K} a^j_1 + \prod_{j=1}^{K} a^j_2 &= a^1_1 b_1 + a^1_2 b_2 \\ &\geq a^1_1 b_2 + a^1_2 b_1 \\ &= \prod_{j=1}^{K} a^j_{\psi_j(1)} + \prod_{j=1}^{K} a^2_{\psi_j(2)}. \end{aligned}$$

If, for at least one j (say, $j = 2$), $\psi_j(1) = 1, \psi_j(2) = 2$, then by the induction assumption

$$\begin{aligned} & a^1_1 \left[a^2_1 \prod_{j=3}^{K} a^j_{\psi_j(1)} \right] + \left[a^1_2 \, a^2_2 \prod_{j=3}^{K} a^j_{\psi_j(2)} \right] \\ &= a^1_1 \left[\prod_{j=2}^{K} a^j_{\psi_j(1)} + \prod_{j=2}^{K} a^j_{\psi_j(2)} \right] + (a^1_2 - a^1_1) \left[\prod_{j=2}^{K} a^j_{\psi_j(2)} \right] \\ &\leq a^1_1 \left[\prod_{j=2}^{K} a^j_1 + \prod_{j=2}^{k} a^j_2 \right] + (a^1_2 - a^1_1) \left[\prod_{j=2}^{K} a^j_{\psi_j(2)} \right] \end{aligned}$$

$$\leq a_1^1 \left[\prod_{j=2}^{K} a_1^j + \prod_{j=2}^{K} a_2^j \right] + (a_2^1 - a_1^1) \prod_{j=2}^{K} a_2^j$$

$$= \prod_{j=1}^{K} a_1^j + \prod_{j=1}^{K} a_2^j .$$

Hence the inequality holds for $n = 2$, and all integer values of k.

Returning to arbitrary n, if $\psi_2,\ldots,\psi_k$ are not all equal to ψ_1 then there exist two values of i (say $i = 1$, and $i = 2$) such that $\psi_j(1) \leq \psi_j(2)$ does not hold for all $j = 2,\ldots,k$. However, by considering the permutations ψ_j' $(j = 2,\ldots,k)$ which are the same as ψ_j, $(j = 2,\ldots,k)$ except that $\psi_j'(1) \leq \psi_j'(2)$ for every $j = 2,\ldots,k$ it follows from the above result for $n = 2$ that (keeping $\psi_1'(i) = i$)

$$\sum_{i=1}^{n} \prod_{j=1}^{k} a^j_{\psi_j'(i)} \geq \sum_{i=1}^{n} \prod_{j=1}^{k} a^j_{\psi_j(i)}.$$

Hence, the original permutations could not be optimal.

Now let $R(x_1,\ldots,x_n)$ be any real-valued joint cumulative probability distribution function. For our purposes we want to prove the following extension.

Theorem 1:[1] If $a_1^j \leq \ldots \leq a_n^j$ $(j = 1,\ldots,k)$, then

$$\sum_{i=1}^{n} R(a_i^1,\ldots,a_i^k) \geq \sum_{i=1}^{n} R(a^1_{\psi_1(i)}, a^2_{\psi_2(i)},\ldots,a^k_{\psi_k(i)}),$$

where $\psi_1(i) = i$ $(i = 1,\ldots,n)$ and ψ_j $(j = 2,\ldots,k)$ are any permutations of $1, 2,\ldots,n$.

---

1. This extension turns out to be a rediscovery of a special case of a result obtained by Lorentz [2].

Proof: As in the above proof we need only prove the theorem for $n = 2$. Also, without loss of generality we can then take $a_1^j = 0$, $a_2^j = 1$, $j = 1,\dots,k$, and we can take R to be a discrete distribution function with mass only at the points $X_1,\dots,X_{2^k}$ where X_m $(m = 1,\dots,2^k)$ are all the points consisting of the k coordinates which are 0 or 1. Let C_m denote the probability mass at the point X_m. For any $\psi_2,\dots,\psi_k$ we can write

$$R(a^1_{\psi_1}(i),\dots,a^k_{\psi_k}(i)) = \sum_{m=1}^{2^k} C_m \prod_{j=1}^{k} g_{mj}(a^j_{\psi_j}(i)) ,$$

where

$$\begin{aligned} g_{mj}(a) &= 1, \text{ if } a \geq x_{mj} \\ &= 0, \text{ if } a < x_{mj} \end{aligned}$$

where x_{mj} is the j^{th} coordinate of X_m.

This is so since

$$\begin{aligned} \prod_{j=1}^{k} g_{mj}(a^j_{\psi_j}(i)) &= 0, \quad \text{if } a^j_{\psi_j}(i) < x_{mj} \text{ for at least one } j \\ &= 1, \quad \text{if } a^j_{\psi_j}(i) \geq x_{mj} \text{ for all } j = 1,\dots,k. \end{aligned}$$

But since $g_{mj}(a)$ are non-decreasing functions it follows from the Hardy, Littlewood, and Polya result for products that

$$\prod_{j=1}^{k} g_{mj}(a_1^j) + \prod_{j=1}^{k} g_{mj}(a_2^j) \geq \prod_{j=1}^{k} g_{mj}(a^j_{\psi_j}(1)) + \prod_{j=1}^{k} g_{mj}(a^j_{\psi_j}(2))$$

for any permutations $\psi_2,\dots,\psi_k$. Then, since $c_m \geq 0$ $(m = 1,\dots,2^k)$

it follows that

$$R(a_1^1,\ldots,a_1^k) + R(a_2^1,\ldots,a_2^k) \geq R(a^1_{\psi_1(1)},\ldots,a^k_{\psi_k(1)})$$
$$+ R(a^1_{\psi_1(1)},\ldots,a^k_{\psi_k(2)})$$

for every $\psi_2,\ldots,\psi_k$ as was to be shown.

2. Application to Reliability Theory

We are concerned with a type of system that has k components. Associated with each component is a numerical value. Let $\{a^j\}$ $(j = 1,\ldots,k)$ denote the set of numerical values of the k components. We assume that $R(a^1,\ldots,a^k)$ is the probability that the system will perform satisfactorily (i.e. $R(a^1,\ldots,a^k)$ is the reliability of the system) where $R(a^1,\ldots,a^k)$ as a function of $a^1,\ldots,a^k$ has the properties of a joint cumulative distribution function. For example, if the system's performance depends on values of k random variables, $Y_1,\ldots,Y_k$, to the extent that the system performs satisfactorily, if and only if, $Y_j \leq a^j$, $j = 1,2,\ldots,k$, then if $Y_1,\ldots,Y_k$ have joint distribution $F(y_1,\ldots,y_k)$, then $R(a^1,\ldots,a^k) = F(a^1,\ldots,a^k)$ will have the assumed interpretation and properties.

Now suppose $a_1^j \leq \ldots \leq a_n^j$ are n components of type j $(j = 1,\ldots,k)$. Then n systems can be assembled from these components. Let N denote the number of systems that perform satisfactorily. N is a random variable whose distribution will depend on the way the n systems are assembled. A direct application of theorem 1 asserts:

Theorem 2: Of all the $(n!)^{k-1}$ different ways in which the n systems can be assembled, EN is maximized if these n systems have reliabilities $R(a_i^1,\ldots,a_i^k)$ $(i = 1,\ldots,n)$.

Proof. For any assembly defined by $\psi_1(i) = i$, $\psi_j(i)$ $(j = 2,\ldots,n)$

$$EN = \sum_{i=1}^{n} R(a^1_{\psi_1(i)}, a^2_{\psi_2(i)},\ldots,a^k_{\psi_k(i)}).$$

The previous result holds.

We can assert, also, the following result.

Theorem 3: If $R(a^1_{\psi_1(i)},\ldots,a^k_{\psi_k(i)}) \geq 1/2$ for every i and $\psi_2,\ldots,\psi_k$ then the same assembly that maximizes EN minimizes the variance of N.

Proof. Since variance $N = \sum_{i=1}^{n} R(a^1_{\psi_1(i)},\ldots,a^k_{\psi_k(i)})(1-R(a^1_{\psi_1(i)},\ldots,a^k_{\psi_k(i)}))$ the truth of the theorem need only be established for $n = 2$. That it is true for $n = 2$ follows from the following lemma:

Lemma: If $1/2 \leq p_1 \leq q_1 \leq p_2 \leq 1$, $p_1 \leq q_2 \leq p_2$, and $p_1+p_2 \geq q_1+q_2$ then $p_1(1-p_1) + p_2(1-p_2) \leq q_1(1-q_1) + q_2(1-q_2)$.

Proof: On subtracting the right hand member from the left we get

$$p_1(1-p_1) + p_2(1-p_2) - q_1(1-q_1) - q_2(1-q_2)$$

$$= p_1 + p_2 - p_1^2 - p_2^2 - q_1 - q_2 + q_1^2 + q_2^2$$

$$= q_1^2 - p_1^2 + q_2^2 - p_2^2 + (p_1 - q_1) + (p_2 - q_2)$$

$$= (q_1 - p_1)(q_1 + p_1) + (q_2 - p_2)(q_2 + p_2) - (q_1 - p_1) - (q_2 - p_2)$$

$$= (q_1 - p_1)[q_1 + p_1 - 1] + (q_2 - p_2)(q_2 + p_2 - 1)$$

$$\leq (p_2 - q_2)(q_1 + p_1 - 1) + (q_2 - p_2)(q_2 + p_2 - 1)$$

$$= (p_2 - q_2)(q_1 + p_1 - 1) - (p_2 - q_2)(q_2 + p_2 - 1)$$

$$= (p_2 - q_2)(q_1 + p_1 - q_2 - p_2)$$

$$= -(p_2 - q_2)(q_2 - q_1 + p_2 - p_1) \leq 0.$$

Some remarks are in order:

1. Theorems 2 and 3 will be applicable to special coherent structures, namely, those where the reliability function is not only monotone but also a distribution function.

2. It should be stressed that the exact form of the distribution function need not be known, only that the reliability function can be assumed to be in the form of some distribution function.

[3] Another Application of the Hardy, Littlewood and Polya Result To Reliability

Consider the following model: A stockpile consists of n units of each of k different types of items. Associated with each item is a probability that the item will perform effectively. We denote by P_i^j, $j = 1,\ldots,k$, $i = 1,\ldots,n$, the probability that the i^{th} unit of the type j will perform effectively. (This item will be referred to as the P_i^j item.) These probabilities are assumed to be independent and it is supposed that

$$P_1^j \leq P_2^j \leq \ldots \leq P_n^j, \quad j = 1,\ldots,k.$$

From these nk items we must construct n modules, where a module consists of one of each of the k types of items. We say that a module functions if all of its k elements perform effectively. The problem is to construct the modules in such a way so as to maximize the probability of attaining at least $0 \leq r \leq n$ functioning modules.

Let us define the i^{th} module to be that module containing the item P_i^1. If our objective was to maximize the expected number of functioning modules, then the Hardy, Littlewood and Polya result yields the solution. Namely, that the i^{th} module consists of the items $P_i^1, P_i^2,\ldots,P_i^k$. We now show that this construction not only maximizes the expected number of functioning modules but it also possesses the stronger property of maximizing the probability that the number of functioning modules is at least r, for each $0 \leq r \leq n$.

Theorem 4: The probability of obtaining at least r functioning modules is maximized, for each $r = 1,\ldots,n$, by letting the i^{th} module consist of the items

$$P_i^1,\ P_i^2,\ldots,P_i^k,\quad i = 1,\ldots,n.$$

Proof: Consider any arbitrary construction of modules - call it C_1 - whose 1st module consists of the items

$$P_1^1,\ P_{i_2}^2,\ P_{i_3}^3,\ldots,P_{i_k}^k,$$

where $i_2 \neq 1$, and suppose that the sth module of C_1 consists of the items

$$P_s^1,\ P_1^2,\ P_{j_3}^3,\ldots,P_{j_k}^k.$$

Now consider a different construction - call it C_2 - whose 1st module consists of the items

$$P_1^1,\ P_1^2,\ P_{\ell_3}^3,\ldots,P_{\ell_k}^k$$

and whose sth module consists of the items

$$P_s^1,\ P_{i_2}^2,\ P_{m_3}^3,\ldots,P_{m_k}^k,$$

where

$$\ell_r = \min(i_r,j_r),\ m_r = \max(i_r,j_r),\ r = 3,\ldots,k.$$

Suppose further that the remaining modules at C_2 are identical to those at C_1. The probability that either the 1st or the sth module

of C_1 functions is

$$P^1_1 . P^2_{i_2} . P^3_{i_3} \cdots\cdots P^k_{i_k} + (1 - P^1_1 . P^2_{i_2} . P^3_{i_3} \cdots\cdots P^k_{i_k})(P^1_s . P^2_1 . P^3_{j_3} \cdots\cdots P^k_{j_k})$$

while the corresponding probability under C_2 is

$$P^1_1 . P^2_1 . P^3_{\ell_3} \cdots\cdots P^k_{\ell_k} + (1 - P^1_1 . P^2_1 . P^3_{\ell_3} \cdots\cdots P^k_{\ell_k})(P^1_s . P^2_{i_2} . P^3_{m_3} \cdots\cdots P^k_{m_k}) .$$

As the number of the other $n - 2$ modules which function is stochastically equal under C_1 or C_2 it follows by the Hardy, Littlewood and Polya result that the number of functioning modules is stochastically larger under C_2 than it is under C_1. Hence we need only consider constructions whose first module contains both P^1_1 and P^2_1. Similarly we can show that we need only consider constructions whose first module is $P^1_1, P^2_1, \ldots, P^k_1$. Repeating this argument on the other modules completes the proof.

We make the following remarks:

1. The main reason we have considered this problem is that often the "total system" works if, and only if, at least r of the modules function.

2. If we suppose that $k = 2$ and that a module functions if at least one of the items in the module performs effectively then a proof similar to the above shows that the construction stochastically maximizing the number of functioning components is the one whose i^{th} module consists of the items P^1_i, P^2_{n+1-i}.

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